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Note

The 3-choosability of plane graphs of girth 4 ☆

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Abstract

A set S of vertices of the graph G is called *k-reducible* if the following is true: G is k-choosable if and only if $G-S$ is k-choosable. A *k-reduced subgraph* H of G is a subgraph of G such that H contains no k-reducible set of some specific forms. In this paper, we show that a 3-reduced subgraph of a non-3-choosable plane graph G contains either adjacent 5-faces, or an adjacent 4-face and k-face, where $k \leqslant 6$. Using this result, we obtain some sufficient conditions for a plane graph to be 3-choosable. In particular, if G is of girth 4 and contains no 5- and 6-cycles, then G is 3-choosable.

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1. Introduction

In this paper, we consider only finite and simple graphs. Undefined terms may be found in [2]. Suppose k is an integer. Then k^+ and k^- denote integers $\geqslant k$ and $\leqslant k$, respectively. A vertex u is called a *k-vertex* if $d_G(u) = k$. If $k = 3$, then u is called a *minor* vertex. A face f is called a *k-face* if $d_G(f) = k$. If $k = 4$, then f is called a *minor* face. If no confusion can arise, $d(v)$ and $d(f)$ will be used instead of $d_G(v)$ and $d_G(f)$, respectively. A face of a plane graph is *incident* with all edges and vertices on its boundary. Two faces are *adjacent*

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if they have an edge in common. Let F_j be the set of j-faces incident with j minor vertices and F_j^i be the subset of F_j consisting only of faces adjacent to i minor faces.

A *k-cycle* is a cycle of length k. The name of a cycle will also be used to denote the set of its vertices. We use $F(u)$ and $k(u)$ to denote the set of all faces incident with the vertex u and $\min\{d(x)|x \in F(u)\}$, respectively.

A *list coloring* of G is an assignment of colors to $V(G)$ such that each vertex v receives a color from a prescribed list $L(v)$ of colors and adjacent vertices receive distinct colors (see [7]). $L(G)=(L(v)|v \in V(G))$ is called a *color list* of G. The graph G is called *k-choosable* if G admits a list coloring for all color lists L with k colors in each list.

All 2-choosable graphs have been characterized by Erdős et al. [3]. In [5], Thomassen proved that every plane graph is 5-choosable. From that proof, one can find a simple linear algorithm for finding the corresponding list coloring. Voigt [8] showed that there are planar graphs which are not 4-choosable. It remains to decide whether a given plane graph is 4- or 3-choosable. Gutner [4] proved that these problems are NP-hard. So far, some sufficient conditions have been obtained and some constructions have been found. Alon and Tarsi [1] proved that every plane bipartite graph is 3-choosable. Thomassen [6] proved that every plane graph of girth at least 5 is 3-choosable. Voigt [9] gave an example of a non-3-choosable plane graph of girth 4. The objective of this paper is to study 3-choosability of plane graphs of girth 4. In this paper we give a necessary condition for a plane graph of girth 4 to be 3-choosable. In particular, if G is of girth 4 and contains no 5- and 6-cycles, then G is 3-choosable.

2. Some lemmas

Let C be a cycle in G. A *sub-cycle* of C is a cycle in $G[C]$ with length strictly shorter than C. We shall need some lemmas, the first of which is well known.

Lemma 1. *Let v be a vertex of G, not necessarily planar, with $d(v)\leqslant k-1$. Then G is k-choosable if and only if $G-\{v\}$ is k-choosable.*

Lemma 2. *Let G be a graph having no subgraphs isomorphic to K_4. Suppose $n\geqslant 2$ and C is a $2n$-cycle of G and $d(u)\leqslant k$ for all $u \in C$. Then either* (i) *there exists an even sub-cycle C^* of C and with $d(u)\leqslant k$ for all $u \in C^*$ or* (ii) *G is k-choosable if and only if $G-C$ is k-choosable.*

Proof. Suppose that $G-C$ is k-choosable and suppose that $L=(L(v)|v \in V(G))$ is a color list of G in which each list contains k colors. Let ϕ be an L-list coloring of $G-C$. For all $v \in C$, let $L^0(v)=L(v)\backslash\{\phi(u)|u \in V(G)\backslash C$ and $vu \in E(G)\}$. Then $|L^0(v)|\geqslant 2$. We have three cases:

Case 1: $G[C]$ is chordless. Since even cycles are 2-choosable, there exists a 2-list coloring ϕ' of C. Combining ϕ and ϕ', we obtain an L-list coloring of G.

Case 2: $G[C]$ contains exactly one chord. Without loss of generality, assume that $C= u_1u_2\cdots u_{2n}$ and u_1u_j is the chord, where $3\leqslant j\leqslant 2n-1$. Then $|L^0(u_1)|\geqslant 3$, $|L^0(u_j)|\geqslant 3$ and $|L^0(u_i)|\geqslant 2$ if $i\neq 1$ or j. We can then define ϕ' on C as follows: choose $\phi'(u_j) \in L^0(u_j)\backslash L^*$, where L^* is a subset of $L^0(u_{j-1})$ with $|L^*|=2$, $\phi'(u_i) \in L^0(u_i)\backslash\{\phi'(u_{i-1})\}$ for

$i = j+1, j+2, \ldots, 2n$, $\phi'(u_1) \in L^0(u_1)\backslash\{\phi'(u_{2n}), \phi'(u_j)\}$, $\phi'(u_i) \in L^0(u_i)\backslash\{\phi'(u_{i-1})\}$ for $i = 2, \ldots, j-2$ (if $j \geqslant 4$) and $\phi'(u_{j-1}) \in L^*\backslash\{\phi'(u_{j-2})\}$. Combining ϕ and ϕ', we obtain an L-list coloring of G.

Case 3: $G[C]$ has two or more chords. If $n=2$, then $G[C]$ is isomorphic to K_4, contrary to the conditions of the lemma. So we may assume that $n \geqslant 3$. Suppose u_1u_i and u_ju_k are the two chords. If $3 \leqslant i \leqslant j < k-1 \leqslant 2n-1$, then $u_1u_2\cdots u_i$, $u_1u_iu_{i+1}\cdots u_ju_ku_{k+1}\cdots u_{2n}$ and $u_ku_ju_{j+1}\cdots u_{k-1}$ are also three cycles with total length of $2n+4$. Therefore, one of these cycles must be even. Since the length of each cycle must be at least 3, length of each of these cycles is at most $2n-2$. If $2 \leqslant j < i < k \leqslant 2n$, then the length of the cycles $C_1 = u_1u_2\cdots u_i$, $C_2 = u_ju_{j+1}\cdots u_k$ and $C_3 = u_1\cdots u_ju_ku_{k-1}\cdots u_i$ are i, $k-j+1$ and $j+k-i+1$, respectively. The total length of C_1, C_2 and C_3 is therefore even. If either C_1 or C_2 is even, then we have obtained an even sub-cycle of C. If both C_1 and C_2 are odd, then C_3 is even. If $j+k-i+1 = 2n$, then $C_4 = u_1u_iu_{i-1}\cdots u_ju_ku_{k+1}\cdots u_{2n}$ is of length $2n-k+j-i+3 = 4$. Therefore either C_3 or C_4 is an even sub-cycle of C. □

A set S of vertices of graph G is called *k-reducible* if the following is true: G is k-choosable if and only if $G-S$ is k-choosable. Hence by Lemmas 1 and 2, if $\delta(G) \leqslant k-1$, or if G has an even cycle passing through k^--vertices, then G contains a k-reducible subset. A *k-reduced* subgraph of G is obtained from G by deleting successively k-reducible sets as determined by Lemmas 1 and 2 until no such set exists. If G contains no k-reducible sets as determined by Lemmas 1 and 2, we say that G is 3-*reduced*. The following lemma follows from definition.

Lemma 3. *G is k-choosable if and only if a k-reduced subgraph of G is k-choosable.*

By Lemma 3, in order to determine whether a graph G is 3-choosable or not, it is sufficient to consider the choosability of a 3-reduced subgraph of G. The following lemma can be obtained from definition and Lemma 2.

Lemma 4. *If H is a k-reduced subgraph of G and H does not contain K_4's, then any even cycle of H contains a $(k+1)^+$-vertex.*

3. Main results

Theorem 5. *Let G be a non-3-choosable plane graph of girth not less than 4 and let H be any 3-reduced subgraph of G. Then there exists in H adjacent 5-faces or adjacent 4-face and k-face ($k \leqslant 6$).*

Corollary 6. *Let G be a plane graph of girth not less than 4. If a 3-reduced subgraph of G contains neither adjacent 5-faces nor adjacent 4-face and k-face ($k \leqslant 6$), then G is 3-choosable.*

Corollary 7. *Let G be a plane graph of girth not less than 4. If G contains neither adjacent 5-cycles nor adjacent 4-cycle and k-cycle ($k \leqslant 6$), then G is 3-choosable.*

Clearly, graphs satisfying conditions of Corollary 7 may contain 4-, 5- or 6-cycles.

Corollary 8. *Let G be a plane graph of girth not less than* 4. *If G contains no* 5- *and* 6-*cycles, then G is* 3-*choosable*.

Proof. Let G be a plane graph without 5- and 6-cycles. Consider a 3-reduced subgraph H of G. Certainly $\delta(H) \geqslant 3$ and H contains no 5- and 6-cycles. Hence, H contains neither adjacent 5-faces nor adjacent 4-face and k-face ($k \leqslant 6$). It follows from Corollary 6 that H is 3-choosable. Thus, the corollary holds by Lemma 3. $\square$

4. Proof of Theorem 5

Let G be a non-3-choosable plane graph. Without loss of generality, we assume that G is 3-reduced. Suppose the theorem is false. Then there exist neither adjacent 5-faces nor adjacent 4-face and k-face with $k \leqslant 6$. That is, there are no adjacent 5-faces and $d(f) \geqslant 7$ for every face f adjacent to a 4-face. This also implies that $F_7 = F_7^0 \cup F_7^1 \cup F_7^2 \cup F_7^3$. If we assign a weight of $\sigma(x) = d(x) - 4$ to each $x \in V(H) \cup F(H)$, then by Euler's formula we have:

$$\sum_{v \in V(H) \cup F(H)} \sigma(x) = -8. \tag{1}$$

If we obtain a new weight $\sigma^*(x)$ for each $x \in V(H) \cup F(H)$ by transferring weights from one element to another, then we also have

$$\sum_{v \in V(H) \cup F(H)} \sigma^*(x) = -8. \tag{2}$$

Moreover, if $\sigma^*(x) \geqslant 0$ for all $x \in V(H) \cup F(H)$, then the theorem is proved.

Weights will be transferred from $f \in F$ to $v \in V$, where v is incident with f, $d(f) \geqslant 5$ and $d(v) = 3$, according to the following rules:

(R1) Transfer $\frac{1}{2}$ if (a) $d(f) \geqslant 7$ and $f \notin F_7$ or (b) $f \in F_7$ and $k(v) = 4$.
(R2) Transfer $\frac{2}{5}$ if (a) $d(f) = 6$ or (b) $f \in F_7$ and $k(v) = 5$.
(R3) Transfer $\frac{1}{5}$ if (a) $d(f) = 5$ or (b) $f \in F_7$ and $k(v) = 6$.

Let $v \in V$. If $d(v) \geqslant 4$, then $\sigma^*(v) = \sigma(v) \geqslant 0$. Suppose $d(v) = 3$, v is incident with f_1, f_2 and f_3 and $d(f_1) \leqslant d(f_2) \leqslant d(f_3)$. By Lemma 4, at most one of these three faces is in F_7. If $d(f_1) = 4$, then (R1) is applicable to both f_2 and f_3. Therefore, $\sigma^*(v) = \sigma(v) + 2 \cdot \frac{1}{2} = 0$. If $d(f_1) = 5$, then (R3(a)) is applicable to f_1 and (R3) is not applicable to f_2 and f_3. So $\sigma^*(v) \geqslant \sigma(v) + \frac{1}{5} + 2 \cdot \frac{2}{5} = 0$. If $d(f_1) = 6$, then (R3(b)) is applicable to at most one of f_2 and f_3, and (R3(a)) is not applicable. So $\sigma^*(v) \geqslant \sigma(v) + \frac{1}{5} + 2 \cdot \frac{2}{5} = 0$. If $d(f_1) \geqslant 7$, then (R1(a)) is applicable to at least two of f_1, f_2 and f_3. So $\sigma^*(v) \geqslant \sigma(v) + 2 \cdot \frac{1}{2} = 0$.

Let $f \in F$. If $d(f) = 4$, then $\sigma^*(f) = \sigma(f) = 0$. If $d(f) = 5$, then $\sigma^*(f) \geqslant \sigma(f) - 5 \cdot \frac{1}{5} = 0$. If $d(f) = 6$, then at least one 4^+-vertex is incident with f by Lemma 4. So $\sigma^*(f) \geqslant$

$\sigma(f) - 5 \cdot \frac{2}{5} = 0$. If $d(f) = k \geqslant 8$, then $\sigma^*(f) \geqslant \sigma(f) - k \cdot \frac{1}{2} \geqslant k - 4 - (k/2) \geqslant 0$. If $d(f) = 7$ and $f \notin F_7$, then at least one 4^+-vertex is incident with f. So $\sigma^*(f) \geqslant \sigma(f) - 6 \cdot \frac{1}{2} = 0$.

If $f \in F_7^0 \cup F_7^1$, then weights are transferred from f to at most two 3-vertices according to [R1(b)]. So $\sigma^*(f) \geqslant \sigma(f) - 2 \cdot \frac{1}{2} - 5 \cdot \frac{2}{5} = 0$.

If $f \in F_7^2$ and f is not adjacent to any 6-face, then at least one vertex incident with f, say v, is incident with 7^+-faces only, and 0 is transferred from f to v. So $\sigma^*(f) \geqslant \sigma(f) - 4 \cdot \frac{1}{2} - 2 \cdot \frac{2}{5} > 0$. If $f \in F_7^2$ and f is adjacent to a 6-face f_1, then at least one face adjacent to both f and f_1 is a 7^+-face. So $\sigma^*(f) \geqslant \sigma(f) - 4 \cdot \frac{1}{2} - 2 \cdot \frac{2}{5} - \frac{1}{5} = 0$.

If $f \in F_7^3$, then one vertex incident with f is incident with 7^+-faces only, and the weight transferred from f to v is 0. So $\sigma^*(f) \geqslant \sigma(f) - 6 \cdot \frac{1}{2} = 0$.

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